Gray optical dips in the subpicosecond regime

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Narrow optical dip solutions are investigated when, besides self-phase modulation and group velocity dispersion, also third-order dispersion, nonlinear dispersion, and stimulated Raman scattering are taken into account. By using the inverse scattering transform for the higher-order optical nonlinear Schrödinger (HNLS) equation under Hirota parameter conditions, the dark *N*-soliton solution is constructed. The explicit forms of the one- and two-soliton solutions are investigated in detail. The results show an interesting property of the gray two-soliton solution. Two gray dips do not interact provided their modulation depths are appropriately chosen. In addition, when generalizing the HNLS equation (to regions beyond the Hirota parameter conditions), it can be shown that also quite stable generalized two-dip solitary wave solutions exist. The latter, although not belonging to integrable systems, approximately preserve most of the interesting properties detected for the integrable Hirota equation.

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I. INTRODUCTION

Since the invention of the optical soliton [1,2] in the early seventies, optical soliton physics turned out to be one of the fastest growing fields in the modern science. The solitary light waves are of particular interest in optical fiber systems because of their enormous potential for telecommunication and ultrafast signal-routing systems [3]. The concept has demonstrated already a huge potential for applications, and it promises fundamental progress in basic research as well. For example, bright solitons are well established in optical telecommunication; dark solitons may play a new interesting role in guiding light by light.

From the theoretical point of view, the success of optical solitons is based on the properties of the one-dimensional cubic nonlinear Schrödinger (NLS) equation as the generic and robust model. The NLS equation can be used for investigating pulses in the picosecond regime. Obviously, narrower optical solitary pulses are candidates for higher bit rates. Also the faster switching is supported by smaller widths. Thus, during the last years very short pulses became a topic of growing interest. In the subpicosecond regime ($\leqslant$100 fs), higher-order effects, such as third-order dispersion (TOD), self-steepening, and self-frequency shift become important [4]. Considering these effects, Kodama and Hasegawa [5] derived a higher-order optical nonlinear Schrödinger (HNLS) equation. Subsequently, many authors have analyzed the HNLS equation, preferentially under zero boundary conditions and from different points of view. They obtained new exact solutions, such as optical shocks and the bright *N*-soliton solution [6–16].

In contrast to the bright solitary wave solutions, the characteristics of dark solutions of the HNLS equation (under nonzero boundary conditions) is less known. In the picosecond regime, dark solitary waves or solitons follow from the NLS equation. They have been studied by several authors [1,17–25] in both, theory and experiment. In recent years, the femtosecond dark solitary waves or solitons became of increasing interest. The black solitary wave solution and the so-called combined solitary wave solutions for the HNLS equation under nonzero boundary condition have been found [8,26–28]. Very recently, Mahalingam and Porsezian [28] investigated the integrability of the HNLS equation by employing a Painlevé analysis. They constructed an explicit Lax pair for the HNLS equation under specific conditions for its coefficients. In that case the HNLS equation is called Hirota equation. The Hirota bilinear form was used to generate the dark (black) one and two solitons. The gray *N*-soliton solution of the Hirota equation has not been reported yet. As we shall demonstrate, the inverse scattering transform (IST) can be used to construct new solutions that show interesting properties in itself.

In this paper, we work out the details of the IST for the Hirota equation under nonzero boundary conditions. The formalism will be used to construct the (general) dark (gray) *N*-soliton solution. The dark one- and two-soliton solutions will be presented in explicit forms. Interesting physical applications will arise from the characteristics of the gray two-soliton solution. After discussing the peculiarities of the latter, we pose the question whether the new phenomena will also prevail in the nonintegrable regime. For that we extend the model beyond the Hirota conditions, i.e., we consider the HNLS equation without further restrictions of its coefficients. We shall obtain a (generalized) dark (gray) solitary wave solution. Numerically, we shall demonstrate that also in the nonintegrable regime noninteracting gray two-dip solutions exist.

The manuscript is organized as follows. In Sec. II, we briefly present the general model. In Sec. III the specific Hirota conditions are assumed. The IST formalism will be worked out in detail in order to use it for constructing general solutions. The general dark *N*-soliton solution follows in Sec. IV for reflectionless potentials. Special attention is given to the gray two-soliton solution. After a generalization to the nonintegrable case in Sec. V, the paper is concluded in Sec. VI by a short summary and discussion.

II. THE MODEL

The standard model for localized ultrashort light waves in the subpicosecond (or femtosecond) regime is the $(1+1)$-dimensional HNLS equation. The latter describes the propagation of an optical mode in Z-direction. When the mode is localized, we use its typical length in time T for normalization. Linear as well as nonlinear group velocity dispersion (GVD), the Kerr nonlinearity [via self-phase modulation (SPM)], and stimulated Raman scattering (SRS) are taken into account. In a frame moving with the group velocity, the HNLS equation is written as [5]

$$E_Z = i(\alpha_1 E_{TT} + \alpha_2 |E|^2 E) + \alpha_3 E_{TTT} + \alpha_4 (|E|^2 E)_T + \alpha_5 E(|E|^2)_T, \tag{1}$$

where E is the slowly varying envelope of the electric field, the subscripts Z and T denote the spatial and temporal partial derivatives in retarded time coordinates, respectively, and α_1, α_2, α_3, α_4, and α_5 are coefficients following from GVD, SPM, TOD, self-steepening, and self-frequency shift (arising from SRS), respectively.

For picosecond light pulses, the last three terms on the right-hand side of Eq. (1) can be omitted, and Eq. (1) reduces to the NLS equation. The NLS equation includes only the linear GVD and the SPM. It admits bright or dark soliton-type pulse propagation in the anomalous or normal dispersion regimes, respectively [1,2,29,30]. The NLS equation is the generic model for envelope solitons. In numereous cases, it has been used to describe the dominating balance between nonlinearity and dispersion to produce stable localized solutions. However, as has been mentioned already, for ultrashort light pulses, whose duration is shorter than 100 fs, the last three terms on the right-hand side of Eq. (1) are important and should be retained.

By employing appropriate scaling transformations, Eq. (1) can be reduced to a two-parameter canonical form. In the following, however, we shall keep the original formulation with five parameters in order to better identify the roles of the various physical effects.

Searching for general solutions, let us first apply a Galilean transform [16]

$$E(Z,T) = E_0 q(Z, T - Z/V) \exp[i(KZ - \Omega T)], \tag{2}$$

where V is the group velocity shift, Ω is the frequency shift, and K represents the phase shift. An additional parameter E_0 has been introduced here (it will be fixed later) in order to simplify some notations.

After this transformation, the function $q(z,t)$ (with arguments $z=Z$ and $t=T-Z/V$) satisfies

$$q_z + i\beta_0 q + \beta_1 q_t - i\beta_2 q_{tt} - \beta_3 q_{ttt} - i\beta_n |q|^2 q - \beta_4 |q|^2 q_t - \beta_5 q(|q|^2)_t = 0, \tag{3}$$

where $\beta_0 = K + \alpha_1\Omega^2 - \alpha_3\Omega^3$, $\beta_1 = -V^{-1} - 2\alpha_1\Omega + 3\alpha_3\Omega^2$, $\beta_2 = \alpha_1 - 3\alpha_3\Omega$, $\beta_n = |E_0|^2(\alpha_2 - \alpha_4\Omega)$, $\beta_3 = \alpha_3$, $\beta_4 = \alpha_4|E_0|^2$, and $\beta_5 = (\alpha_4 + \alpha_5)|E_0|^2$. Equation (3) is the starting point for the following investigation.

III. THE IST METHOD FOR THE HIROTA EQUATION

Generally, Eq. (3) is not integrable, except for some special cases. The latter have been identified by several authors, and the original ideas of Hirota [6] as well as Sasa and Satsuma [7] have been confirmed. In the following, we shall concentrate on the Hirota equation. The latter is obtained from the HNLS equation (1) when the parameters satisfy the conditions:

$$\alpha_4 + \alpha_5 = 0, \tag{4}$$

$$\alpha_1\alpha_4 - 3\alpha_2\alpha_3 = 0. \tag{5}$$

We shall call these restrictions Hirota conditions (constraints). Hirota [6] has presented exact envelope solutions under the conditions mentioned above. He also showed that the solutions reveal the close relation between classical solitons and envelope solitons. Subsequently, many authors discussed the integrability of the Hirota equation, including such interesting topics as the Painlevé test, Lax pairs, boundary conditions, and so on.

The Hirota equation can be solved by the IST method. In the following, we shall work out the details of that procedure for nonzero boundary conditions. The idea is to use that formalism to construct gray soliton (dip) solutions. Under Hirota conditions, by setting $|E_0|^2 = -6\alpha_3/\alpha_4$, $K = -\alpha_1\Omega^2 + \alpha_3\Omega^3 + \mu^2|E_0|^2(\alpha_2 - \alpha_4\Omega)$ and employing the 2×2 AKNS method [31], the Lax pair for Eq. (3) follows from

$$\mathbf{\Psi}_t = \mathbf{L}\mathbf{\Psi}, \tag{6}$$

$$\mathbf{\Psi}_z = \mathbf{M}\mathbf{\Psi}, \tag{7}$$

where

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{8}$$

in the form

$$\mathbf{L} = \begin{pmatrix} -i\lambda & q \\ q^* & i\lambda \end{pmatrix}, \tag{9}$$

$$\mathbf{M} = i(b_1\lambda + b_2\lambda^2 + b_3\lambda^3)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - (b_1 + b_2\lambda + b_3\lambda^2) \times \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix} + \frac{1}{2} i(b_2 + b_3\lambda)\begin{pmatrix} |q|^2 - \mu^2 & -q_t \\ q_t^* & -|q|^2 + \mu^2 \end{pmatrix} - \frac{1}{4} b_3 \begin{pmatrix} q^* q_t - q q_t^* & 2|q|^2 q - 2\mu^2 q - q_{tt} \\ 2|q|^2 q^* - 2\mu^2 q^* - q_{tt}^* & q q_t^* - q^* q_t \end{pmatrix}. \tag{10}$$

Here, $b_1 = -V^{-1} - 2\alpha_1\Omega + 3\alpha_3\Omega^2 + 2\mu^2\alpha_3$, $b_2 = -2(\alpha_1 - 3\Omega\alpha_3)$, $b_3 = 4\alpha_3$, and λ is the spectral parameter. Considering nonzero boundary conditions for dark solitons, we have introduced a positive constant μ^2 denoting the asymptotic value of the dark N-soliton intensity $|q|^2$ as time

t approaches infinity (i.e., $|q|^2 \to \mu^2$ as $|t| \to \infty$). Using the compatibility condition $\mathbf{L}_z - \mathbf{M}_t + [\mathbf{L},\mathbf{M}] = 0$, one can easily derive Eq. (3) under the Hirota conditions. The present Lax pair is basically the same as that presented in Ref. [28]. In the following, it will turn out that the present form is more convenient to solve the Lax equations (6) and (7) under nonzero boundary conditions.

The construction of the Lax pair suggests that Eq. (3) is integrable under nonzero boundary condition. However, the existence of a Lax pair does not always ensure to solve Eqs. (6) and (7) in an explicit form. Therefore, the general N-soliton solution under nonzero boundary condition still requires to work out the details of the IST.

Having in mind that a dark soliton solution may exist with different asymptotic behaviors in phase at $t \to \pm\infty$, without the loss of generality we write the nonzero boundary conditions as $q \to \mu$ as $t \to +\infty$, and $q \to \mu e^{i\phi}$ as $t \to -\infty$. Here, ϕ is an arbitrary constant to allow for phase differences. From the asymptotic behavior of the operator L [see Eq. (9)],

$$L \to L_0 = -i\lambda\sigma_3 + U_0 \quad \text{as } t \to +\infty, \tag{11}$$

$$L \to L_- = -i\lambda\sigma_3 + U_- \quad \text{as } t \to -\infty, \tag{12}$$

where

$$U_0 = \mu\sigma_1, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

and

$$U_- = Q(\beta) U_0 Q^{-1}(\beta), \quad Q(\beta) = e^{i\beta\sigma_3/2}, \tag{14}$$

we find that in the limit $t \to +\infty$, the Lax equation (6) reduces to

$$E_t(t,\lambda) = L_0 E(t,\lambda). \tag{15}$$

Its solution is

$$E(t,\lambda) = \begin{pmatrix} 1 & -i\mu^{-1}(\lambda-\kappa) \\ i\mu^{-1}(\lambda-\kappa) & 1 \end{pmatrix} e^{-i\kappa t\sigma_3} \quad \text{for real numbers } \lambda \text{ and } \lambda^2 \geqslant \mu^2, \tag{16}$$

where $\kappa = \sqrt{\lambda^2 - \mu^2}$.

In the opposite limit $t \to -\infty$, we have

$$E_{-,t}(t,\lambda) = L_- E_-(t,\lambda) \tag{17}$$

with the solution

$$E_-(x,t) = Q(\beta) E(t,\lambda). \tag{18}$$

Generally, κ is a double-valued function of λ. By employing an auxiliary parameter ζ instead of introducing a Riemann surface, we rewrite the parameters λ and κ [32],

$$\lambda = \frac{1}{2}(\zeta + \mu^2 \zeta^{-1}), \tag{19}$$

$$\kappa = \frac{1}{2}(\zeta - \mu^2 \zeta^{-1}). \tag{20}$$

It is easy to verify that for real ζ the parameters λ and κ are also real, with $\lambda^2 \geqslant \mu^2$. The Jost solutions $\Psi(t,\zeta)$ and $\Phi(t,\zeta)$ for Eq. (6) can be expressed as

$$\Psi(t,\zeta) = E(t,\zeta) + \int_t^\infty K(t,s) E(s,\zeta) ds \tag{21}$$

and

$$\Phi(t,\zeta) = E_-(t,\zeta) + \int_{-\infty}^t K(t,s) E_-(s,\zeta) ds. \tag{22}$$

Here, the kernel function $K(t,s)$ is a 2×2 matrix, being independent of ζ. It satisfies

$$K(t,t) - \sigma_3 K(t,t) \sigma_3 + U(t) - U_0 = 0, \tag{23}$$

$$K_t(t,s) + \sigma_3 K_s(t,s)\sigma_3 - U(t)K(t,s) + \sigma_3 K(t,s) \sigma_3 U_0 = 0 \quad \text{for } t < s, \tag{24}$$

$$K(t,s) = 0 \quad \text{for } t > s, \quad K(t,+\infty) = 0, \tag{25}$$

and

$$K(t,s) = \sigma_1 \overline{K(t,s)} \sigma_1. \tag{26}$$

The scattering data for the operator L form the set $s = \{r(\zeta); \zeta_j, c_j, j = 1,2,\dots,n\}$, where $r(\zeta) = b(\zeta)/a(\zeta)$ is the reflection coefficient with $|a(\zeta)|^2 - |b(\zeta)|^2 = 1$, $a(\zeta)$ can be analytically continued to the upper-half plane $\mathrm{Im}\zeta \geqslant 0$, ζ_j are the discrete eigenvalues determined by the zeros of $a(\zeta)$ with $|\zeta_j| = \mu$ and $\mathrm{Im}\zeta_j > 0$, c_j are the asymptotic characteristics of the functions. Within the IST, we find that the scattering data $s(z)$ evolve according to

$$a(z,\zeta) = a(\zeta), \tag{27}$$

$$b(z,\zeta) = b(\zeta) e^{-2i\kappa(b_1 + b_2\lambda + b_3\lambda^2)z}, \tag{28}$$

$$r(z,\zeta) = r(\zeta) e^{-2i\kappa(b_1 + b_2\lambda + b_3\lambda^2)z}, \tag{29}$$

$$\zeta_j(z) = \zeta_j(0) = \zeta_j, \tag{30}$$

$$c_j(z) = c_j e^{-2i\kappa_j(b_1 + b_2\lambda_j + b_3\lambda_j^2)z}. \tag{31}$$

The canonical formalism of the IST enables us to obtain the Gel'fand-Levitan-Marchenko integral equation for reconstruction of the kernel $K(t,s;z)$ from the scattering data $s(z)$. We have

$$K(t,s;z) + F(t+s;z) + \int_t^{+\infty} K(t,\tau;z) F(\tau+s;z) d\tau = 0 \quad \text{for } t < s, \tag{32}$$

where

$$F(t;z)=\frac{1}{2}\sum_{j=1}^{n}c_j(z)\begin{pmatrix}\mu\\ i\zeta_j\end{pmatrix}(1-i\mu\zeta_j^{-1})e^{i\kappa_j t}+\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{r(z,\zeta)}{i\zeta}\begin{pmatrix}\mu\\ i\zeta\end{pmatrix}(1-i\mu\zeta^{-1})e^{i\kappa t}d\zeta. \tag{33}$$

From the Eqs. (23) and (26) we find the solution $q(t,z)$ of Eq. (3) in the form

$$q(t,z)=\mu-2K_{12}(t,t;z). \tag{34}$$

This concludes the explicit solution of the initial-value problem.

IV. *N*-SOLITON SOLUTIONS

In principle, we could now follow the general solution strategy prescribed by the IST to solve any well-posed initial-value problem in *z*. Starting with an arbitrary initial distribution in *t*, during the dynamical evolution in *z*, solitons and radiation may appear. Here, we concentrate on the simpler problem of determining only the possible soliton solutions. As is known from other cases, solitons correspond to the discrete spectrum. Thus, when we start with certain types of initial distributions leading only to the discrete spectra at $z=0$, we can expect to determine the soliton solutions directly.

A. General expressions for reflectionless potentials

For the case of reflectionless potentials, i.e., for $r(z,\zeta)=0$, the Gel'fand-Levitan-Marchenko equation (32) reduces to

$$0=K(t,s;z)+\frac{1}{2}\sum_{j=1}^{n}c_j(z)\begin{pmatrix}\mu\\ i\zeta_j\end{pmatrix}(1-i\mu\zeta_j^{-1})e^{i\kappa_j(t+s)}+\int_{t}^{+\infty}K(t,\tau;z)\frac{1}{2}\sum_{j=1}^{n}c_j(z)\begin{pmatrix}\mu\\ i\zeta_j\end{pmatrix}(1-i\mu\zeta_j^{-1})e^{i\kappa_j(\tau+s)}d\tau. \tag{35}$$

As is well known, one can solve that integral equation with the ansatz

$$K_{\mu\nu}(t,s;z)=\sum_{m=1}^{n}c_m(z)A_{\mu\nu,m}(t)e^{i\kappa_m s},\qquad \mu,\nu=1,2. \tag{36}$$

A straightforward calculation leads to

$$\sum_{m=1}^{n}\left(\delta_{jm}+\frac{\mu c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}\right)A_{11,m}(t)+\sum_{m=1}^{n}\frac{i\zeta_j c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}A_{12,m}(t)=-\frac{1}{2}\mu e^{-\eta_j t}, \tag{37}$$

$$\sum_{m=1}^{n}\left(\frac{-i\bar{\zeta}_j c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}\right)A_{11,m}(t)+\sum_{m=1}^{n}\left(\delta_{jm}+\frac{\mu c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}\right)A_{12,m}(t)=\frac{1}{2}i\bar{\zeta}_j e^{-\eta_j t}, \tag{38}$$

$$\sum_{m=1}^{n}\left(\delta_{jm}+\frac{\mu c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}\right)A_{21,m}(t)+\sum_{m=1}^{n}\left(\frac{i\zeta_j c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}\right)A_{22,m}(t)=-\frac{1}{2}i\zeta_j e^{-\eta_j t}, \tag{39}$$

$$\sum_{m=1}^{n}\left(\frac{-i\bar{\zeta}_j c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}\right)A_{21,m}(t)+\sum_{m=1}^{n}\left(\delta_{jm}+\frac{\mu c_m(z)e^{-(\eta_m+\eta_j)t}}{2(\eta_m+\eta_j)}\right)A_{22,m}(t)=-\frac{1}{2}\mu e^{-\eta_j t}, \tag{40}$$

for $j=1,\ 2,\dots,\ n$, where $\zeta_m=\xi_m+i\eta_m$, $|\zeta_m|=\mu$, and $\eta_m=-i\kappa_m$, for $m=1,\ 2,\cdots,\ n$.

The complex conjugates of the Eqs. (37) and (38) are consistent with Eqs. (39) and (40). Therefore, in the following we need only to proceed with Eqs. (37) and (38). This is a set of algebraic equations which can be written in matrix form,

$$(I+\mu B)\tilde{A}_{11}+JB\tilde{A}_{12}=-\frac{1}{2}\mu f, \tag{41}$$

$$\bar{J}B\tilde{A}_{11}+(I+\mu B)\tilde{A}_{12}=-\frac{1}{2}\bar{J}f, \tag{42}$$

where

$$B=(B_{jm})_{n\times n},\tilde{A}_{11}=(\tilde{A}_{11,m})_{n\times 1},\tilde{A}_{12}=(\tilde{A}_{12,m})_{n\times 1},\quad J=i\ \mathrm{diag}(\zeta_1,\zeta_2,\dots,\zeta_n). \tag{43}$$

The matrix components are

$$f_j=\sqrt{c_j(z)}e^{-\eta_j t},\qquad B_{jm}=\frac{f_j f_m}{2(\eta_m+\eta_j)}, \tag{44}$$

$$\tilde{A}_{11,m} = \sqrt{c_m(z)} A_{11,m}(t), \quad \tilde{A}_{12,m} = \sqrt{c_m(z)} A_{12,m}(t). \tag{45}$$

Solving these equations systematically, we arrive at

$$\tilde{A}_{12} = -\frac{1}{2}\mu^2 (J + \mu JB + \mu BJ)^{-1} f. \tag{46}$$

Substituting the solution (46) into Eq. (36), we can evaluate the right-hand side of Eq. (34) to obtain the dark N-soliton solution in the general form

$$q_n(t,z) = \mu \frac{\det(J + \mu JB + \mu BJ + \mu f f^T)}{\det(J + \mu JB + \mu BJ)}. \tag{47}$$

For $n=1$ this simplifies to the dark (gray) one-soliton solution

$$q_1(t,z) = \frac{\xi_1 - i\eta_1}{\mu} (\xi_1 + i\eta_1 \tanh[\eta_1 \theta_1(t,z)]), \tag{48}$$

where $\mu^2 = \xi_1^2 + \eta_1^2$ with $\eta_1 > 0$, and $\theta_1(t,z) = t - t_1 - (b_1 + b_2 \xi_1 + b_3 \xi_1^2) z$ with $\mu c_1 / 2\eta_1 = e^{-2t_1}$. The solution possesses the asymptotic behaviors

$$q_1(t,z) \to \mu \ \text{as}\ t \to +\infty, \tag{49}$$

$$q_1(t,z) \to \mu \frac{\overline{\zeta_1}}{\zeta_1} \ \text{as}\ t \to -\infty. \tag{50}$$

The amplitude of the dark one soliton is given by $|q_1(t,z)| = \{(\xi_1^2 + \eta_1^2) - \eta_1^2 \mathrm{sech}^2[\eta_1 \theta_1(t,z)]\}^{1/2}$. The minimum and maximum amplitudes are ξ_1 and $\sqrt{\xi_1^2 + \eta_1^2}$, respectively. It should be pointed out that the dark one-soliton solution (48) is more general than that one presented in Ref. [28] by the Hirota method. The latter solution is a black soliton solution, for which the minimum of the amplitude $|q_1(t,z)|$ is zero. The present solution (48) includes gray solitons,

B. Properties of a gray two-soliton solution

For $n=2$, Eq. (47) leads to the explicit form of a general dark two soliton

$$q_2(t,z) = \mu \left[1 + \frac{4\mu(q_{11}(t,z) + q_{12}(t,z) - 2\mu) - 4i \dfrac{\zeta_1 + \zeta_2}{\eta_1 + \eta_2} (q_{11}(t,z) - \mu)(q_{12}(t,z) - \mu)}{4\mu^2 + \left(\dfrac{\zeta_1 + \zeta_2}{\eta_1 + \eta_2} \right)^2 (q_{11}(t,z) - \mu)(q_{12}(t,z) - \mu)} \right]. \tag{51}$$

Here, $q_{11}(t,z) \equiv q_1(t,z;\zeta_1)$ and $q_{12}(t,z) \equiv q_1(t,z;\zeta_2)$ are dark one-soliton solutions with parameters ζ_1 and ζ_2 [see Eq. (48)], respectively. The asymptotic behaviors are

$$q_2(t,z) \to \mu \ \text{as}\ t \to +\infty, \tag{52}$$

$$q_2(t,z) \to \mu \left(\frac{\overline{\zeta_2} + \overline{\zeta_1}}{\zeta_2 + \zeta_1} \right)^2 \ \text{as}\ t \to -\infty. \tag{53}$$

Although not strictly valid, a two-soliton solution may be interpreted as the superposition of two interacting (effective) one-soliton solutions approaching each other from infinity. This interpretation leads to scenarios for soliton-soliton interactions. The latter are important issues, e.g., for estimating bit-error rates (BER). In the following, we may therefore draw interesting conclusions by inspecting details of the two-soliton solution (51). From the latter, we can clearly recognize that, similar to the characteristics of dark solitons in the picosecond domain, the femtosecond dark soliton solutions retain their shapes after interaction. Only slight changes in their phases occur. The interaction of dark solitons is elastic. Figure 1 shows graphically an interaction derived from the dark two-soliton solution (51). From this plot, we conclude that the interaction force between the two dark (one) solitons in the femtosecond domain is effectively repulsive. Since there is no possibility to form a bound state, this is an important feature in favor of dark solitons when compared to bright solitons. Because of the repulsive interaction, in the future dark solitons may become of interest also in long-distance communication systems.

It is important to note that it is possible to construct a two-soliton solution that effectively does not show any interaction between the dips, i.e., the distance between the two (effective) one-soliton solutions does not change during propagation in z. This happens if the modulation depths satisfy $\xi_2 = -(\xi_1 + b_2/b_3)$. Figure 2 presents such an evolution of a gray two-soliton solution. The plot shows that the distance between the two dips remains unchanged if the modulation depths are chosen appropriately. This property is, e.g., very interesting for optical communication with low BER or for guiding light by light.

V. THE GENERAL MODEL AND ITS SOLITARY SOLUTIONS

In the preceding section, we investigated the N-soliton solution under Hirota conditions (4) and (5). However, the Hirota conditions may not be satisfied in real wave guides.

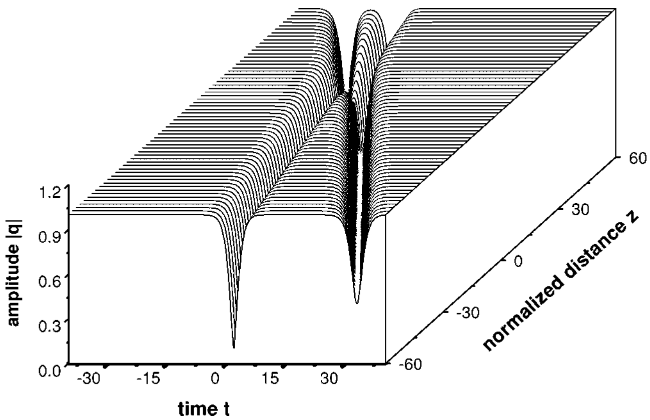

FIG. 1. Plot of the dark two-soliton solution (51). The parameter values are $t_1=2.5$, $t_2=-2.5$, $b_1=0$, $b_2=-1.0$, $b_3=1.0$, $\mu=1.0$, $\xi_1=0.4$, and $\xi_2=0.1$; η_1 and η_2 are determined from $\mu^2=\xi_1^2+\eta_1^2=\xi_2^2+\eta_2^2$, respectively.

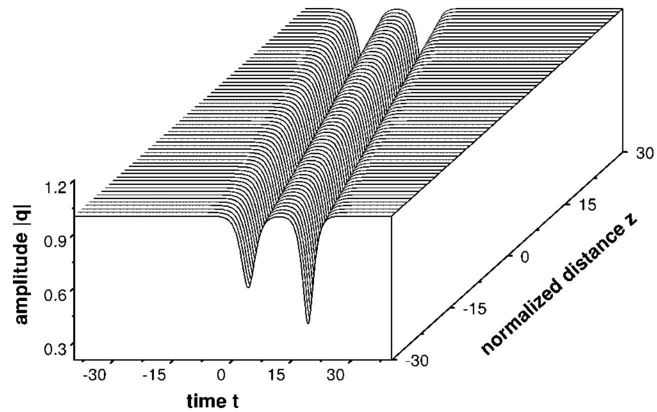

FIG. 2. Plot of the dark two-soliton solution (51). The parameter values are the same as in Fig. 1 except for $\xi_2=-(\xi_1+b_2/b_3)=0.6$.

One could therefore argue that it will be difficult to observe the proposed solutions in the experiment. As we shall demonstrate, fortunately the Hirota constraints (4) and (5) are not needed for the existence of interesting solutions. In the following, we shall construct dark solitary dip solutions for the general HNLS equation, and not only for the (integrable) Hirota equation. Thus, now we return to the HNLS equation without the Hirota constraints. We shall present dark solitary dip solutions in explicit forms.

In order to proceed, we make an ansatz similar to the (integrable) dark one-soliton solution (48),

$$q(z,t)=e^{i\phi}\{\lambda+i\rho\tanh[\eta(t-\chi z)]\}. \tag{54}$$

Here, ϕ is introduced to take care of a possible constant phase, η and χ are the pulse width and the inverse group velocity shift, respectively, and finally λ designates the depth of modulation of the dark solitary wave. The solitary wave amplitude is given by $|q(z,t)|=\{(\lambda^2+\rho^2)-\rho^2\mathrm{sech}^2[\eta(t-\chi z)]\}^{1/2}$; the minimum and maximum amplitudes are λ and $\sqrt{\lambda^2+\rho^2}$, respectively.

Substituting the ansatz (54) into Eq. (3) and equating the coefficients of independent terms, we find that the parameters have to satisfy

$$6\beta_3\eta^2+(\beta_4+2\beta_5)\rho^2=0, \tag{55}$$

$$2\beta_2\eta^2+2\beta_5\lambda\eta\rho+\beta_n\rho^2=0, \tag{56}$$

$$\eta\chi-\beta_1\eta-\rho\lambda\beta_n-2\beta_3\eta^3+\beta_4\eta\lambda^2=0, \tag{57}$$

$$\beta_0-\beta_n(\lambda^2+\rho^2)=0. \tag{58}$$

For the reason of simplicity, we set $E_0=1$. Next, we substitute for β_j the expressions in terms of α_k. These algebraic manipulations lead to four equations being equivalent to Eqs. (55)–(58).

At first we observe that the second equation (56) is identically satisfied under the Hirota constraints (4) and (5). Thus, the integrable case will allow for one additional free parameter compared to the situation with $(\alpha_4+\alpha_5)\neq 0$. In any case, we shall consider η and Ω as free parameters. In the Hirota case, in addition, λ can be freely chosen.

Under the condition $\alpha_3(3\alpha_4+2\alpha_5)<0$ and for $(\alpha_4+\alpha_5)\neq 0$, we determine the parameters ρ, λ, χ, and K as

$$\rho=\eta\sqrt{\frac{-6\alpha_3}{3\alpha_4+2\alpha_5}}, \tag{59}$$

$$\lambda=\sqrt{\frac{-6\alpha_3}{3\alpha_4+2\alpha_5}}\left(\Omega-\frac{\alpha_1(3\alpha_4+2\alpha_5)-3\alpha_2\alpha_3}{6\alpha_3(\alpha_4+\alpha_5)}\right), \tag{60}$$

$$\chi=(-2\alpha_1\Omega+3\alpha_3\Omega^2)+2\alpha_3\eta^2+(\alpha_2-\alpha_4\Omega)\lambda\rho/\eta-\alpha_4\lambda^2, \tag{61}$$

$$K=-\alpha_1\Omega^2+\alpha_3\Omega^3+(\alpha_2-\alpha_4\Omega)(\lambda^2+\rho^2). \tag{62}$$

From Eq. (59), we recognize that the condition for a dark solitary wave solution (54) is

$$\alpha_3(3\alpha_4+2\alpha_5)<0. \tag{63}$$

If we force the frequency shift Ω to satisfy

$$\Omega=\frac{\alpha_1(3\alpha_4+2\alpha_5)-3\alpha_2\alpha_3}{6\alpha_3(\alpha_4+\alpha_5)}, \tag{64}$$

the dark solitary wave solution (54) becomes a black one. The latter is consistent with the dark solitary wave solution presented in Ref. [26] (setting $\alpha_1=\alpha_2=\alpha_3=1$, $\alpha_4=c_1$, and $\alpha_5=c_2$).

For the Hirota equation, i.e. when $\alpha_4+\alpha_5=0$ and $\alpha_1\alpha_4=3\alpha_2\alpha_3$, the solution (60) is no longer valid. Instead, λ becomes a free parameter. Equation (59) now reads

$$\rho^2=-\frac{6\alpha_3}{\alpha_4}\eta^2. \tag{65}$$

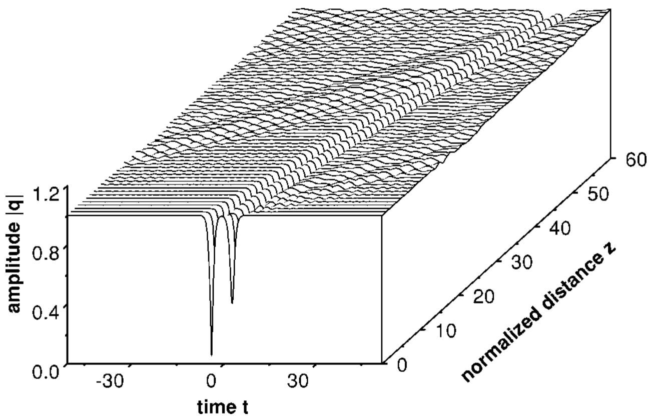

FIG. 3. Plot of the evolution of a dark two-dip solitary solution in the nonintegrable case. Compared to Fig. 2, we have modified the parameters α_4 and α_5 by 10% and used $\xi_1=0.4$, $\xi_2=0.0545$, $\Omega=-0.9697$, $b_1=0.2355$, $b_2=-0.4545$, and $b_3=1.0$. Otherwise, the parameter values are the same as in Fig. 2.

For χ and K we can use Eqs. (61) and (62), respectively, taking into account the Hirota constraints (4) and (5). These results agree with the solution (48) constructed by the IST for reflectionless potentials.

Besides the free parameter λ, for the Hirota equation another point is worth mentioning. The existence condition (63) simplifies for Eq. (4) to

$$\alpha_3\alpha_4<0. \tag{66}$$

On the other hand, making use of the second condition (5), one finds

$$\alpha_1\alpha_2<0. \tag{67}$$

This condition agrees with the condition for dark solitons known from the (integrable) NLS equation. It is worth mentioning that the latter inequality is not necessary for the solutions (59)–(62). However, we expect that for $\alpha_1\alpha_2>0$ the solutions become unstable.

So far nothing is known about the stability of the general solitary dip solutions when the Hirota conditions (4) and (5) are not satisfied. The question of robustness, together with the investigation of a general solitary N-dip solution, requires plenty of numerical simulations. Here we only demonstrate on one example that, when deviating from the Hirota constraints (4) and (5), the above mentioned interaction properties are still (approximately) true. In Fig. 3, we have modified the parameters α_4 and α_5 compared to those shown in Fig. 2 (α_4 has been increased by 10% and α_5 has been decreased by 10%); they do not fulfill anymore the Hirota constraints. Then we start a simulation with an initial two-dip solution where the modulation depths haven been chosen appropriately. No principal differences to Fig. 2 occur, except for an enhanced radiation. More simulation results will be presented separately, together with the form as with the well as structural stability considerations. By the latter, we mean that the (general) HNLS equation can be further generalized by taking into account additional terms describing, e.g., filters, amplification, attenuation, etc. Then the behavior of the dips in the presence of perturbations, the existence of stable attractors, and the characterization of the basins of attraction are new questions to be discussed elsewhere.

VI. SUMMARY AND DISCUSSION

In summary, in the first part we have solved the Hirota equation by employing the IST method and obtained the dark N-soliton solution. The one- and two-soliton solutions have been presented in explicit forms. The interaction of dark solitons was discussed. These results show that there is an interesting property of the gray two-soliton solution. The distance between the two gray solitons remains unchanged with the normalized distance, provided the modulation depths are appropriately chosen. This property may be used in optical communication systems to increase the bit rate. In the second part, we have investigated the general HNLS equation. Again, a generalized dark solitary wave solution exists. The analytical form of a general dark N-soliton solution is still unknown. Numerically, we have shown that also for the general case interesting gray two-soliton solutions exist, which (approximately) possess similar behaviors as the integrable ones.

It should be noted that the condition for the existence of femtosecond *dark* solitary wave solutions is $\alpha_3(3\alpha_4+2\alpha_5)<0$. The existence criterion depends on the properties of third-order dispersion, self-steepening, and self-frequency shift. This result complements previous findings. It was shown in Ref. [14] that femtosecond *bright* solitary wave solutions exist for $\alpha_3(3\alpha_4+2\alpha_5)>0$. As the value of $3\alpha_4+2\alpha_5$ is generally negative, it means that femtosecond dark (bright) solitary waves can be realized in the positive (negative) TOD regime.

Finally, we should remind the reader that there appears another constraint for the existence of dark solitary waves in the practical systems. In fact, the frequency shift Ω should be much smaller than the carrier frequency ω_0 to maintain the validity of the model (3) under the quasimonochromatic approximation. This requirement can be satisfied in the normal group velocity dispersion regime ($\alpha_1<0$). For demonstration, let us take a typical single mode optical fiber as an example. If one chooses the optical fiber parameters as [4]: $\alpha_1=-10\ \mathrm{ps}^2/\mathrm{km}$, $\alpha_2=20\ \mathrm{W}^{-1}/\mathrm{km}$, and $\alpha_3=0.02\ \mathrm{ps}^3/\mathrm{km}$, one can estimate $\Omega\sim4\times10^{-2}\omega_0$ from Eq. (64) for black optical pulses of 100 fs with wavelength 1.06 μm. Therefore, the femtosecond dark solitary wave should exist in optical fibers with positive TOD in the normal group velocity dispersion regime.

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